rings rather than disks have been examined and the results show that all three models can be utilized in determination of this orientation. However, no unambiguous indication of which is the best function to use emerges. This may indeed mean that all models used, which by their nature are gross approximations to the actual structure under study, suffer equally from this approximation, with local variations in particular calculations favouring one or the other. The reasonable results obtained using the 'selective' hoop and ring models in the simulated cases suggest that further pursuance of the location of several planar groups simultaneously and selectively (on bond lengths/interatomic vector distributions) may be fruitful and work is continuing on this possible development of the ideas advanced here.

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Full Tables of Colour Space Groups with Colour-Preserving Translations

BY J. N. KOTZEV AND D. A. ALEXANDROVA

Faculty of Physics, Sofia University, 5 Anton Ivanov Blvd, Sofia BG-1126, Bulgaria

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Abstract

All 2571 permutational colour space groups $G^{(P)} \equiv T^{(1)}\hat{G}^{(P)}$ isostructural to the 230 space groups are tabulated and classified. The method for constructing these groups is described. The permutational representation $D_G^{H'}$ associated with each $G^{(P)}$ is given in order to make the physical applications easier. The group-theoretical criteria of the Landau theory have been checked in connection with the colour space-group application in phase transition analysis. Comparison with the results of other authors is given.

1. Introduction

The theory of crystallographic colour groups is a relatively new group-theoretical approach in the description of the structure and physical properties of crystals (see Shubnikov & Koptsik, 1974). Many problems of solid-state physics connected with the determination of the relationships between the symmetry group of the crystal, its subgroups, factor groups and their representations can effectively be solved using the colour-group theory and the corresponding tables of groups. The interpretation of the Landau theory of continuous phase transitions in terms of the permutational colour groups can be given as an example. It is based on the results of Koptsik & Kotzev (1974*a*, *b*), and Kotzev (1975) and is published in a series of papers (Kotzev, Litvin & Birman, 1982; Litvin, Kotzev & Birman, 1982; Kotzev, Koptsik & Rustamov, 1983).

A historical review of the colour-symmetry theory is given in the monograph of Shubnikov & Koptsik (1974) and the recent article by Schwarzenberger (1984). Therein a comprehensive list of the most important papers in this field (including the works of Heesch, Shubnikov, Belov, Wondratschek, van der Waerden & Burckhardt, Zamorzaev *etc.*) can be found. The most general theory covering all possible kinds of generalized (colour) groups was advanced by Koptsik & Kotzev (1974*a*, *b*) (see also Koptsik, 1975; Kotzev, 1975, 1980).

The groups to be discussed in the present paper are a special type of colour group, the *P*-type permutational colour groups, or 'colour groups' for short.

Unfortunately, the derivation and tabulation of all the colour groups is too complicated. The number of the colour *point* groups is finite and all these groups have been derived and published (*e.g.* Shubnikov & Koptsik, 1974; Koptsik & Kotzev, 1974*a*; Harker, 1976; Litvin, Kotzev & Birman, 1982). The colour *space* groups can be subdivided into two types: (i) groups with colour-preserving translation subgroup $T^{(1)}$ – their number is finite and all of them have been derived by Kotzev & Alexandrova (1986); and (ii) groups $T^{(n_1)}\hat{G}^{(n_2)}$ with colour translation subgroup $T^{(n_1)}$ containing a maximal colour-preserving one $T^{(1)}$ of index $n_1 \geq 2$. The list of these groups is finite for

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each fixed index number $n = n_1 n_2$ of colours, but n is unlimited. Up to the present moment only the colour space groups with $n = 2, 3, 4, 6$ and separated classes of groups for higher n have been tabulated (see § 3 and Koptsik & Kuzhukeev, 1972; Zamorzaev, Galyarskii & Palistrant, 1978; Harker, 1981; Karpova, 1983).

The colour-group tables for practical applications should satisfy the following conditions: completeness, reliability and convenience for application. In this paper complete tables of the derived 2571 permutational colour space groups $G^{(P)} \equiv T^{(1)}\hat{G}^{(P)}$ isostructural to the 230 space groups and the permutational representations $D_G^{H'}$ associated with them are given. Here the number n of the permutated 'colours' is equal to 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. A short report of the derivation of these groups and their classification into 'chromomorphic' classes has been given by Kotzev & Alexandrova (1986). The possibilities for applications in the phase-transition theory have also been discussed there. It is important to note the completeness of the present tables - they contain *all* possible permutational colour space groups with colour-preserving lattice, isostructural to the 230 space groups. Each item in the tables corresponds to a class of equivalent colour groups in conformity with the equivalence definition accepted by us (see § 3).

The rest of the present paper is organized as follows. For the sake of convenience the main principles of the P-type permutational colour-group theory are reviewed in § 2. The method of deriving all colour space groups with colour-preserving translations is described in § 3. The full list of the 2571 groups $G^{(P)} \equiv T^{(1)}\hat{G}^{(P)}$ is included in Tables 1 to 32.

In a future paper (Kotzev & Alexandrova, 1988) the application of the tables published here to the analysis of structural and magnetic phase transitions will be discussed. A detailed comparison of our results with similar ones of other authors will be given.

2. Permutational colour groups $G^{(P)}$

Let G be a crystallographic group with elements $g \in G$ and P an arbitrary group with elements $p \in P$. The P-type colour crystallographic groups $G^{(P)}$ belonging to the family of the groups G and P have been defined (see Shubnikov & Koptsik, 1974; Koptsik & Kotzev, 1974*a*) as subgroups of the direct product $P \times G$, of which the elements $\langle p; g\rangle$ contain at least once all the elements $p \in P$ and $g \in G$ as left-hand-side and right-hand-side components, respectively:

$$G^{(P)} = \{\langle p; g\rangle | \text{all } p \in P, \text{ all } g \in G\} \subseteq P \times G. \quad (1)$$

From a crystallographic and physical point of view the colour groups $G^{(P)}$ isomorphic to G, ξ: $G^{(P)} \leftrightarrow G$, $\xi(\langle p; g\rangle) = g \in G$, called 'minor' by Zamorzaev *et al.* (1978), are most important. We shall call these groups $G^{(P)}$ 'isostructural to G' in order to stress that the same elements $g \in G$ appear in G and $G^{(P)}$ [for example, the four cyclic groups generated by 2, m, $\bar{1}$ and $\langle (12); \bar{1}\rangle$ are isomorphic, but only the groups $\{\bar{1}\}$ and $\{(12); \bar{1}\}$ are isostructural]. The colour groups isostructural to isomorphic crystallographic groups G are isomorphic to each other. In this case there exists a homomorphism π from G onto P:

$$\pi: G \to P, \qquad \text{Ker } \pi = H \lhd G. \quad (2)$$

Owing to the isomorphism ξ there exists a colour-preserving subgroup $H^{(1)} \lhd G^{(P)}$ such that

$$H^{(1)} = \{\langle e_P; h\rangle | \text{identity } e_P \in P, h \in H \lhd G\} \subseteq G^{(P)}, \quad (3)$$

$$G^{(P)}/H^{(1)} \cong G/H \cong P.$$

When P is a transitive subgroup of S_n, a symmetric group of n objects ('colours'), the groups $G^{(P)}$ are called permutational colour groups.

In this paper we shall consider only permutational colour groups isostructural to G and for the sake of brevity we shall call them 'colour groups'. Historically, these groups appeared for the first time as symmetry groups of the general orbit of G whose points are 'coloured' in a definite way by n 'colours' (Shubnikov & Koptsik, 1974). In physical applications of $G^{(P)}$, the colours can represent some physical property of crystals.

Each permutational colour group $G^{(P)}$ contains a subgroup $H'^{(P')}$, the maximal subgroup of $G^{(P)}$, preserving at least one colour unchanged:

$$H'^{(P')} = \{\langle p'; h'\rangle | h' \in H' \subseteq G, p' \in P' = \pi(H') \subseteq P\} \subseteq G^{(P)}, \quad (4)$$

$$H'^{(P')} \cong H' \subseteq G.$$

If H' is an invariant subgroup of G then $H'^{(P')} = H^{(1)}$, *i.e.* it coincides with the maximal colour-preserving subgroup $H^{(1)}$ of $G^{(P)}$. If H' is not an invariant subgroup of G, then

$$\text{Core } H' =: \bigcap_{g \in G} gH'g^{-1} = H \lhd G, \quad (5)$$

where H is the maximal invariant subgroup of G contained in H'. In this case

$$\text{Ker } \pi = H = \text{Core } H', \qquad H \cong H^{(1)} \lhd G^{(P)}, \quad (6)$$

where $H^{(1)}$ is the maximal colour-preserving subgroup of $G^{(P)}$.

Each colour group belonging to the family of G and P, and isostructural to G, can be constructed by pairing each $g_i \in G$ with its image $\pi(g_i) = p_i \in P \subseteq S_n$ (van der Waerden & Burckhardt, 1961):

$$G^{(P)} = \{\langle \pi(g); g\rangle | g \in G, \pi(g) = \pi_G^{H'}(g) \in P \subseteq S_n\}. \quad (7)$$

The set $\{\pi(g)|g\in G\}$ is a transitive permutation representation of G. It can be constructed as a set of permutations of the left cosets $\{g_iH'\}$ of the coset decomposition of G with respect to its subgroup H' of index n:

$$\pi(g)=\pi_G^{H'}(g)=\begin{bmatrix} g_1H' & \dots & g_iH' & \dots & g_nH' \\ gg_1H' & \dots & gg_iH' & \dots & gg_nH' \end{bmatrix}$$
$$=p\in P, \qquad (8)$$
$$P=\text{Image } \pi_G^{H'}.$$

Hence $G^{(P)}$ is uniquely determined by G and its subgroup H'. Therefore the symbol $G(H')$ can be used for identifying the colour groups $G^{(P)}$. However, a lot of useful information for $G^{(P)}$ is missed and $G(H')$ may be misinterpreted as a symbol of a Shubnikov magnetic group $G(H)$ (Bradley & Cracknell, 1972).

It is more convenient to use the 'full' symbol $G/H'/H(A,A')_n$ for $G^{(P)}$ introduced by Koptsik & Kotzev (1974*a*). It contains comprehensive information for the structure of $G^{(P)}$, very useful for classification and physical applications of these groups. Here the transitive permutation group $P\in S_n$ is denoted by the symbol $(A,A')_n$. The abstract groups A and $A'\subset A$ with the property Core $A'=C_1$ are isomorphic to the factor groups:

$$\begin{aligned} A&\cong G^{(P)}/H^{(1)}\cong G/H\cong P,\\ A'&\cong H'^{(P')}/H^{(1)}\cong H'/H\cong P'\subset P. \end{aligned} \qquad (9)$$

Hence, the transitive permutation representation $\pi_A^{A'}$ is a faithful representation of A and it is identical to the group P:

$$\text{Image } \pi_G^{H'}=\text{Image } \pi_A^{A'}=P=(A,A')_n. \qquad (10)$$

The transitive permutation groups $(A,A')_n$ serve as a base of the so-called 'chromomorphic' classification of the colour groups (Koptsik & Kotzev, 1974*a*; Kotzev, Koptsik & Rustamov, 1983). All the colour groups $G/H'/H(A,A')_n$ with the same group of colour permutations $(A,A')_n$ belong to the same chromomorphic class, labelled by the symbol of the group $(A,A')_n$. They can be constructed using the same permutation group $P=(A,A')_n=\text{Image } \pi_A^{A'}$, isomorphic to A.

The transitive permutation representation $\pi_G^{H'}$ can be written in a matrix form, as an $n\times n$ matrix $D_G^{H'}$, where

$$D_G^{H'}(g)_{ij}=\begin{cases}1, & \text{if } gg_jH'=g_iH', \text{ or } g_i^{-1}gg_j\in H'\\ 0, & \text{otherwise.}\end{cases} \qquad (11)$$

Obviously, $D_G^{H'}$ is the representation of G, *induced* by the trivial representation $D_{H'}^1$ of H', *i.e.* $D_G^{H'}=D_{H'}^1\uparrow G$, and also $D_A^{A'}=D_{A'}^1\uparrow A$.

At the same time $D_G^{H'}$ is *engendered* by the representation $D_A^{A'}$ of its factor group $A\cong G/H$, *i.e.* $D_G^{H'}=D_A^{A'}\uparrow\uparrow G$. [The symbol '$\uparrow\uparrow$' has been introduced by Kotzev, Litvin & Birman (1982) for labelling the operation 'engendering' the group representation by a factor-group representation. For the meaning of engendered and induced representations see, for example, Jansen & Boon (1967).] Each irreducible representation D_G^j of G, $D_G^j\in D_G^{H'}$, is engendered by some irreducible representation D_A^j of A, $D_A^j\in D_A^{A'}$, *i.e.* $D_G^j=D_A^j\uparrow\uparrow G$.

The group G can be either a space group (we shall use the symbol G) or a point group $\hat{G}\cong G/T$ where T is the translation subgroup of G. The list of the colour point groups $\hat{G}^{(P)}$, classified in 45 chromomorphic classes, is given by Koptsik & Kotzev (1974*a*) [see also Kotzev, Koptsik & Rustamov (1983)]. In the paper of Litvin, Kotzev & Birman (1982) the list of all 279 colour point groups is given together with the explicit form of their permutational representations $D_{\hat{G}}^{\hat{H}'}$ decomposed to irreducible constituents $D_{\hat{G}}^j$.

As has been shown by Kotzev, Litvin & Birman (1982), the colour groups and their permutational representations can be effectively applied in the Landau theory of phase transitions, in the analysis of tensor fields in crystals (Berenson, Kotzev & Litvin, 1982) *etc.* All the equi-translational phase transitions can be analysed by means of colour space groups with colour-preserving translation subgroup.

3. Colour space groups with colour-preserving translations

3.1. *Description of the groups and their derivation*

The colour space groups $G^{(P)}$ to be discussed in this section and their subgroups $H'^{(P')}$ and $H^{(1)}$ have common maximal translation subgroup $T^{(1)}$, which is colour-preserving:

$$T^{(1)}=\{(e_P;(\hat{1}|t))|(\hat{1}|t)\in T\lhd G\}\lhd G^{(P)}. \qquad (12)$$

In the full symbol $G/H'/H(A,A')_n$ the groups H' and H are equi-translational subgroups of the space group G (with the same translation subgroup $T\lhd G$).

Let the homomorphism φ, Ker $\varphi=T$, map G onto its point group $\hat{G}\cong G/T$. Since H' and H are equi-translational subgroups of G with point groups $\hat{H}'\cong H'/T$ and $\hat{H}\cong H/T$, respectively, the same homomorphism φ maps the space-group chain onto the chain of the corresponding point groups:

$$\begin{array}{ccccc} G & \supset & H' & \rhd & H=\text{Core } H' \\ \downarrow\varphi & & \downarrow\varphi & & \downarrow\varphi \\ \hat{G} & \supset & \hat{H}' & \rhd & \hat{H}=\text{Core } \hat{H}'. \end{array} \qquad (13)$$

Owing to the isomorphism ξ between $G^{(P)}$ and G a similar homomorphism ϕ, Ker $\phi=T^{(1)}$, exists for the colour groups:

$$\begin{array}{ccccc} G^{(P)} & \supset & H'^{(P')} & \rhd & H^{(1)} \\ \phi\downarrow & & \phi\downarrow & & \phi\downarrow \\ \hat{G}^{(P)} & \supset & \hat{H}'^{(P')} & \rhd & \hat{H}^{(1)}, \end{array} \tag{14}$$

where

$$\begin{aligned} \hat{G}^{(P)} &\cong G^{(P)}/T^{(1)} \cong \hat{G}, \\ \hat{H}'^{(P')} &\cong H'^{(P')}/T^{(1)} \cong \hat{H}', \\ \hat{H}^{(1)} &\cong H^{(1)}/T^{(1)} \cong \hat{H}. \end{aligned} \tag{15}$$

The homomorphism ϕ leads to the isomorphisms

$$\begin{aligned} \hat{G}^{(P)}/\hat{H}^{(1)} &\cong G^{(P)}/H^{(1)} \cong A, \\ \hat{H}'^{(P')}/\hat{H}^{(1)} &\cong H'^{(P')}/H^{(1)} \cong A'. \end{aligned} \tag{16}$$

Because of the homomorphism ϕ the 279 colour point groups $\hat{G}^{(P)}$ play the same role for the colour space groups $G^{(P)}$ with colour-preserving translations as the 32 point groups $\hat{G} \cong G/T$ for the 230 space groups G, *i.e.*

$$\begin{array}{ccc} (2571)\ G^{(P)} & \xleftrightarrow{\ \xi\ } & (230)\ G \\ \phi\downarrow & & \downarrow\varphi \\ (279)\ \hat{G}^{(P)} & \longleftrightarrow & (32)\ \hat{G} \end{array} \tag{17}$$

where φ and ϕ are the corresponding homomorphisms (13) and (14). For the sake of convenience we will use the symbol $T^{(1)}\hat{G}^{(P)}$ for this type of colour space groups.

From the isomorphisms (15) and (16) it follows that both the space groups $G^{(P)} \equiv G/H'/H(A, A')_n$ and their point groups $\hat{G}^{(P)} \equiv \hat{G}/\hat{H}'/\hat{H}(A, A')_n$ belong to the same chromomorphic class, one of the 45 classes $(A, A')_n$. This can be represented by the following homomorphism diagram, where Ker $\psi = H^{(1)}$ and Ker $\hat{\psi} = \hat{H}^{(1)}$:

$$\begin{array}{ccc} (2571)\ G^{(P)} & \xrightarrow{\ \ \phi\ \ } & (279)\ \hat{G}^{(P)} \\ & \psi\searrow \quad \swarrow\hat{\psi} & \\ & (45)\ (A, A')_n & \end{array} \tag{18}$$

This diagram is commutative, *i.e.* $\psi = \hat{\psi} \circ \phi$, and represents the following mappings:

$$\psi[G^{(P)}] = \hat{\psi} \circ \phi[G^{(P)}] = \hat{\psi}[\hat{G}^{(P)}] = (A, A')_n. \tag{19}$$

Hence, the 'inverse image' $G^{(P)} = \psi^{-1}[(A, A')_n]$ can be constructed in two steps, $\hat{G}^{(P)} = \hat{\psi}^{-1}[(A, A')_n]$ and $G^{(P)} = \phi^{-1}[\hat{G}^{(P)}]$, respectively. The first step has been realized by Koptsik & Kotzev (1974*a*) [see also the tables of Litvin, Kotzev & Birman (1982)]. The second step is reduced to the following procedure (Kotzev & Alexandrova, 1986).

Each of the 230 space groups is expressed as a union of cosets with respect to the translation subgroup T:

$$G = \bigcup_{i=1}^{|\hat{G}|} Tg_i, \quad g_i = (\hat{g}_i|\tau_i), \tag{20}$$

where τ_i, $i = 1, \ldots, |\hat{G}|$, are the fractional translations connected with the point-group elements $\hat{g}_i \in \hat{G}$. Then each colour point group $\hat{G}^{(P)} = \{\langle p_i; \hat{g}_i\rangle\}$ engenders one and only one colour space group $G^{(P)} = T^{(1)}\hat{G}^{(P)} = \phi^{-1}[\hat{G}^{(P)}]$, isostructural to G. The structure of the colour space group is given by the coset decomposition

$$G^{(P)} = \bigcup_{i=1}^{|\hat{G}|} T^{(1)}\langle p_i; g_i\rangle = \bigcup_{i=1}^{|\hat{G}|} T^{(1)}\langle p_i; (\hat{g}_i|\tau_i)\rangle. \tag{21}$$

Here the elements $(\hat{1}|t) \in T$ are combined with identity $e_P \in (A, A')_n$ to obtain $\langle e_P; (\hat{1}|t)\rangle \in T^{(1)}$. The coset representatives $g_i = (\hat{g}_i|\tau_i)$ in the decomposition (20) are paired with the permutations $p_i \in (A, A')_n$ of the corresponding colour point-group elements $\langle p_i; \hat{g}_i\rangle \in \hat{G}^{(P)}$, in accordance with (19).

Here is the place to discuss one of the main problems of the classification and tabulation of any kind of groups, namely the group-equivalence definition. There are 230 space-group types in respect to the proper (or orientation-preserving) affine equivalence:

$$G_2 \sim G_1 = \alpha G_2 \alpha^{-1}, \quad \alpha \in \mathcal{A}^+. \tag{22}$$

The problem of colour-group equivalence is considerably more complicated. Its definition depends on the presumable applications of these groups (*e.g.* Schwarzenberger, 1984).

For example, Koptsik & Kotzev (1974*a*) proposed a colour-group-equivalence definition (we shall call it 'crystallographic' equivalence), replacing in (22) G_i by $G_i^{(P)}$ and $\mathcal{A}^+$ by $S_n \times \mathcal{A}^+$, and reformulated it as follows: *two colour groups* $G_1^{(P)} \equiv G_1/H_1'/H_1$ *and* $G_2^{(P)} \equiv G_2/H_2'/H_2$ *are equivalent if there exists a proper affine transformation* $\alpha \in \mathcal{A}^+$, *such that*

$$G_2 = \alpha G_1 \alpha^{-1} \text{ and } H_2' = \alpha H_1' \alpha^{-1}, \quad \alpha \in \mathcal{A}^+. \tag{23}$$

If the groups $G_1^{(P)}$ and $G_2^{(P)}$ are isostructural to the same group G, *i.e.* $G_1 = G_2 = G$, then H_1' and H_2' are subgroups of G, conjugated by a proper transformation α^+ of the affine normalizer of $G(\alpha^+ \in N_{\mathcal{A}}(G))$, as well as H_1 and H_2. The crystallographic definition (23) was applied to the Shubnikov groups and to most of the published colour-group tables (see Table 34).

However, for applications in physics it is not the best choice: the permutational representations $D_{G_1}^{H_1'}$ and $D_{G_2}^{H_2'}$ associated with colour groups equivalent by (23) may not be equivalent representations of G. This means that physically different phenomena (*e.g.* two different phase transitions) would be assigned to the same class of colour groups. As is well known, in

Table 1. *Colour groups with* $\hat{G} = C_1$

No.	$C_1^i/H'/H(A, A')_n$	i:	1	${}^s\Gamma_1$
1.1	$C_1^i/C_1^k(C_1, C_1)_1$	k:	1	1†

Table 2. *Colour groups with* $\hat{G} = C_i$

No.	$C_i^i/H'/H(A, A')_n$	i:	1	${}^s\Gamma_1^+$	Γ_1^-
2.1	$C_i^i/C_1^k(C_2, C_1)_2$	k:	1	1	1†
2.2	$C_i^i/C_i^k(C_1, C_1)_1$	k:	1	1†	

Table 3. *Colour groups with* $\hat{G} = C_2$

No.	$C_2^i/H'/H(A, A')_n$	i:	1	2	3	${}^s\Gamma_1$	Γ_2
3.1	$C_2^i/C_1^k(C_2, C_1)_2$	k:	1	1	1	1	1†
3.2	$C_2^i/C_2^k(C_1, C_1)_1$	k:	1	2	3	1†	

Table 4. *Colour groups with* $\hat{G} = C_s$

No.	$C_s^i/H'/H(A, A')_n$	i:	1	2	3	4	${}^s\Gamma_1$	Γ_2
4.1	$C_s^i/C_1^k(C_2, C_1)_2$	k:	1	1	1	1	1	1†
4.2	$C_s^i/C_s^k(C_1, C_1)_1$	k:	1	2	3	4	1†	

Table 5. *Colour groups with* $\hat{G} = C_{2h}$

No.	$C_{2h}^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	${}^s\Gamma_1^+$	Γ_2^+	Γ_1^-	Γ_2^-
5.1	$C_{2h}^i/C_1^k(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	1	1
5.2	$C_{2h}^i/C_s^k(C_2, C_1)_2$	k:	1	1	3	2	2	4	1	.	.	1†
5.3	$C_{2h}^i/C_2^k(C_2, C_1)_2$	k:	1	2	3	1	2	3	1	.	1†	
5.4	$C_{2h}^i/C_i^k(C_2, C_1)_2$	k:	1	1	1	1	1	1	1	1†		
5.5	$C_{2h}^i/C_{2h}^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†			

Table 6. *Colour groups with* $\hat{G} = D_2$

No.	$D_2^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	9	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4
6.1	$D_2^i/C_1^k(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1
6.2*a*	$D_2^i/C_2^{(x)k}(C_2, C_1)_2$	k:	1	1	2	2	3	3	3	3	3	1	.	.	1†
6.2*b*	$D_2^i/C_2^{(z)k}(C_2, C_1)_2$	k:	1	2	1	2	2	1	3	3	3	1	.	1†	
6.2*c*	$D_2^i/C_2^{(y)k}(C_2, C_1)_2$	k:	1	1	2	2	3	3	3	3	3	1	1†		
6.3	$D_2^i/D_2^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	1†			

physics the inequivalent representations are related to different phenomena (*e.g.* different phase transitions, energy levels, selection rules, non-zero tensor components).

For this reason we have adopted a different approach to the problem, similar to that of van der Waerden & Burckhardt (1961) and Litvin, Kotzev & Birman (1982), namely the original crystal structure and its symmetry group G are given and the colour groups isostructural to that G are to be classified. *Two colour groups* $G/H_1'/H_1$ *and* $G/H_2'/H_2$ *isostructural to a fixed space group* G *are equivalent if* H_1' *and* H_2' *are conjugated subgroups of* G, *i.e.*

$$G_1 = G_2 = G, \quad H_2' = gH_1'g^{-1}, \quad g \in G. \qquad (24)$$

We shall call the definition (24) a 'physical' equivalence. Obviously, in this case $H_1 = \text{Core } H_1' = \text{Core } H_2' = H_2 = H$. The equivalence of the colour groups means an equivalence of their permutational representations $D_G^{H'}$. So the symmetry properties of different physical phenomena related to a given structure and transformed by inequivalent permutational representations of its symmetry group G are described by inequivalent colour groups.

For the colour space group with colour-preserving lattice two colour groups $G/H_1'/H_1$ and $G/H_2'/H_2$ are equivalent if and only if their colour point groups $\hat{G}/\hat{H}_1'/\hat{H}_1$ and $\hat{G}/\hat{H}_2'/\hat{H}_2$ are equivalent. This assertion follows directly from the homomorphism φ. It allows us to predict exactly the number of all possible classes of equivalent colour space groups with colour-preserving lattice, isostructural to a given G, and to derive their representatives using the procedure described for constructing these groups.

3.2. *Description of the tables*

The list of the colour space groups $G^{(P)} \equiv T^{(1)}\hat{G}^{(P)}$ is given in Tables 1 to 32. Each table contains the groups $G^{(P)}$ with the same $\hat{G} \cong G^{(P)}/T^{(1)}$. In order to get the most compact form of the tables it is more convenient to use the Schoenflies notation. In this notation the full symbol of the derived colour space groups can be obtained from the corresponding colour point-group symbol $\hat{G}/\hat{H}'/\hat{H}(A, A')_n$ by adding appropriate upper indices i, j, k of the space groups G, H', H, respectively. When $H' = H$, the colour-group symbol is given in a reduced form $G/H(A, A')_n$ and only the indices j and k are sufficient. The upper index i of the space group G is given in the head row of each table. In the column i one can find the corresponding indices j and k, if one

Table 7. *Colour groups with* $\hat{G} = C_{2v}$

No.	$C_{2v}^{i}/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	${}^{s}\Gamma_1$	Γ_2	Γ_3	Γ_4
7.1	$C_{2v}^{i}/C_1^{k}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7.3*a*	$C_{2v}^{i}/C_s^{(x)k}(C_2, C_1)_2$	k:	1	1	2	1	2	2	1	2	2	2	3	3	4	1	2	1	2	3	4	3	4	3	1	.	.	1†
7.2	$C_{2v}^{i}/C_2^{k}(C_2, C_1)_2$	k:	1	2	1	1	2	1	2	1	2	1	1	2	1	3	3	3	3	3	3	3	3	3	1	.	1†	
7.3*b*	$C_{2v}^{i}/C_s^{(y)k}(C_2, C_1)_2$	k:	1	2	2	2	2	2	2	2	2	2	3	4	4	3	3	4	4	3	4	3	4	4	1	1†		
7.4	$C_{2v}^{i}/C_{2v}^{k}(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	1†			

Table 8. *Colour groups with* $\hat{G} = D_{2h}$

No.	$D_{2h}^{i}/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	${}^{s}\Gamma_1^+$	Γ_2^+	Γ_3^+	Γ_4^+	Γ_1^-	Γ_2^-	Γ_3^-	Γ_4^-
8.1	$D_{2h}^{i}/C_1^{k}(D_{2h}, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8.2	$D_{2h}^{i}/C_i^{k}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	.	.	.	.
8.3*a*	$D_{2h}^{i}/C_2^{(y)k}(D_2, C_1)_4$	k:	1	1	1	1	1	2	1	1	2	2	2	2	2	1	2	2	3	3	3	3	3	3	3	3	3	3	3	3	1	1	.	.	1	1	.	.
8.3*b*	$D_{2h}^{i}/C_2^{(z)k}(D_2, C_1)_4$	k:	1	1	1	1	1	1	2	1	1	1	2	1	1	2	2	2	2	2	1	1	1	1	3	3	3	3	3	3	1	.	1	.	1	.	1	
8.3*c*	$D_{2h}^{i}/C_2^{(x)k}(D_2, C_1)_4$	k:	1	1	1	1	2	1	1	2	2	2	1	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	1	.	.	1	1	.	.	1
8.4*a*	$D_{2h}^{i}/C_s^{(y)k}(D_2, C_1)_4$	k:	1	2	2	2	1	2	2	2	2	2	2	2	1	2	2	1	4	4	3	4	3	4	3	4	3	4	4	3	1	1	.	.	.	.	1	1
8.4*b*	$D_{2h}^{i}/C_s^{(z)k}(D_2, C_1)_4$	k:	1	2	1	2	2	2	2	2	1	2	1	1	2	2	2	2	1	2	1	1	2	2	3	4	3	3	4	4	1	.	1	.	.	1	.	1
8.4*c*	$D_{2h}^{i}/C_s^{(x)k}(D_2, C_1)_4$	k:	1	2	2	2	1	2	1	2	2	2	2	2	1	2	2	2	3	3	3	4	3	4	3	4	3	4	4	3	1	.	.	1	.	1	1	.
8.5*a*	$D_{2h}^{i}/C_{2v}^{(x)k}(C_2, C_1)_2$	k:	1	10	4	6	2	6	6	5	2	9	4	7	7	9	5	2	16	17	14	16	15	17	18	19	20	22	21	22	1	.	.	.	.	.	.	1†
8.5*b*	$D_{2h}^{i}/C_{2v}^{(z)k}(C_2, C_1)_2$	k:	1	10	3	8	1	10	7	3	8	3	5	10	1	5	5	7	12	12	11	13	11	13	18	19	20	21	21	20	1	.	.	.	.	.	1†	
8.5*c*	$D_{2h}^{i}/C_{2v}^{(y)k}(C_2, C_1)_2$	k:	1	10	4	6	4	9	4	8	2	9	2	7	7	6	5	9	14	15	14	16	15	17	18	19	20	22	21	22	1	.	.	.	.	1†		
8.6	$D_{2h}^{i}/D_2^{k}(C_2, C_1)_2$	k:	1	1	1	1	2	2	2	2	3	3	3	3	3	3	4	4	5	5	6	6	6	6	7	7	8	8	9	9	1	.	.	.	1†			
8.7*a*	$D_{2h}^{i}/C_{2h}^{(x)k}(C_2, C_1)_2$	k:	1	4	4	4	2	4	1	5	5	5	4	5	2	5	5	5	3	3	3	6	3	6	3	6	3	6	6	3	1	.	.	1†				
8.7*b*	$D_{2h}^{i}/C_{2h}^{(z)k}(C_2, C_1)_2$	k:	1	4	1	4	4	4	5	4	1	4	2	1	4	5	5	5	2	5	1	1	4	4	3	6	3	3	6	6	1	.	1†					
8.7*c*	$D_{2h}^{i}/C_{2h}^{(y)k}(C_2, C_1)_2$	k:	1	4	4	4	1	5	4	4	5	5	5	5	2	4	5	2	6	6	3	6	3	6	3	6	3	6	6	3	1	1†						
8.8	$D_{2h}^{i}/D_{2h}^{k}(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	1†							

Table 9. *Colour groups with* $\hat{G} = C_4$

No.	$C_4^i/H'/H(A,A')_n$	i:	1	2	3	4	5	6	${}^s\Gamma_1$	Γ_2	$(\Gamma_3$	$\Gamma_4)^h$
9.1	$C_4^i/C_1^k(C_4, C_1)_4$	k:	1	1	1	1	1	1	1	1	1†	1†
9.2	$C_4^i/C_2^k(C_2, C_1)_2$	k:	1	2	1	2	3	3	1	1†		
9.3	$C_4^i/C_4^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†			

Table 10. *Colour groups with* $\hat{G} = S_4$

No.	$S_4^i/H'/H(A,A')_n$	i:	1	2	${}^s\Gamma_1$	Γ_2	$(\Gamma_3$	$\Gamma_4)$
10.1	$S_4^i/C_1^k(C_4, C_1)_4$	k:	1	1	1	1	1†	1†
10.2	$S_4^i/C_2^k(C_2, C_1)_2$	k:	1	3	1	1†		
10.3	$S_4^i/S_4^k(C_1, C_1)_1$	k:	1	2	1†			

Table 11. *Colour groups with* $\hat{G} = C_{4h}$

No.	$C_{4h}^i/H'/H(A,A')_n$	i:	1	2	3	4	5	6	${}^s\Gamma_1^+$	Γ_2^+	$(\Gamma_3^+$	$\Gamma_4^+)$	Γ_1^-	Γ_2^-	$(\Gamma_3^-$	$\Gamma_4^-)$
11.1	$C_{4h}^i/C_1^k(C_{4h}, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11.2	$C_{4h}^i/C_2^k(D_2, C_1)_4$	k:	1	1	1	1	3	3	1	1	.	.	1	1	.	.
11.3	$C_{4h}^i/C_s^k(C_4, C_1)_4$	k:	1	1	2	2	3	4	1	1	.	.	.	.	1†	1†
11.4	$C_{4h}^i/C_i(C_4, C_1)_4$	k:	1	1	1	1	1	1	1	1	1†	1†	.	.		
11.5	$C_{4h}^i/S_4^k(C_2, C_1)_2$	k:	1	1	1	1	2	2	1	.	.	.	.	1†		
11.6	$C_{4h}^i/C_4^k(C_2, C_1)_2$	k:	1	3	1	3	5	6	1	.	.	.	1†			
11.7	$C_{4h}^i/C_{2h}^k(C_2, C_1)_2$	k:	1	1	4	4	3	6	1	1†						
11.8	$C_{4h}^i/C_{4h}^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†							

Table 12. *Colour groups with* $\hat{G} = D_4$

No.	$D_4^i/H'/H(A,A')_n$	i:	1	2	3	4	5	6	7	8	9	10	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4	Γ_5^h
12.1	$D_4^i/C_1^k(D_4, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2†
12.2*a*	$D_4^i/C_2^{(x)j}/C_1^1(D_4, C_2')_4$	j:	1	2	1	2	1	2	1	2	3	3	1	.	1	.	1†
12.2*b*	$D_4^i/C_2^{(xy)j}/C_1^1(D_4, C_2')_4$	j:	3	3	3	3	3	3	3	3	3	3	1	.	.	1	1†
12.3	$D_4^i/C_2^{(z)k}(D_2, C_1)_4$	k:	1	1	2	2	1	1	2	2	3	3	1	1	1	1	
12.4*a*	$D_4^i/D_2'^k(C_2, C_1)_2$	k:	6	6	5	5	6	6	5	5	7	7	1	.	.	1†	
12.4*b*	$D_4^i/D_2^k(C_2, C_1)_2$	k:	1	3	2	4	1	3	2	4	8	9	1	.	1†		
12.5	$D_4^i/C_4^k(C_2, C_1)_2$	k:	1	1	2	2	3	3	4	4	5	6	1	1†			
12.6	$D_4^i/D_4^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	1†				

Table 13. *Colour groups with* $\hat{G} = C_{4v}$

No.	$C_{4v}^i/H'/H(A,A')_n$	i:	1	2	3	4	5	6	7	8	9	10	11	12	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4	Γ_5
13.1	$C_{4v}^i/C_1^k(D_4, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2†
13.2*a*	$C_{4v}^i/C_s^{(x)j}/C_1^1(D_4, C_2')_4$	j:	1	2	2	2	2	2	1	2	3	4	3	4	1	.	1	.	1†
13.2*b*	$C_{4v}^i/C_s^{(xy)j}/C_1^1(D_4, C_2')_4$	j:	3	3	3	3	4	4	4	4	3	3	4	4	1	.	.	1	1†
13.3	$C_{4v}^i/C_2^{(z)k}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	3	3	3	3	1	1	1	1	
13.4*a*	$C_{4v}^i/C_{2v}'^k(C_2, C_1)_2$	k:	11	11	11	11	13	13	13	13	18	18	19	19	1	.	.	1†	
13.4*b*	$C_{4v}^i/C_{2v}^k(C_2, C_1)_2$	k:	1	8	3	10	3	10	1	8	20	21	20	21	1	.	1†		
13.5	$C_{4v}^i/C_4^k(C_2, C_1)_2$	k:	1	1	3	3	1	1	3	3	5	5	6	6	1	1†			
13.6	$C_{4v}^i/C_{4v}^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	11	12	1†				

Table 14. *Colour groups with* $\hat{G} = D_{2d}$

No.	$D^{i}_{2d}/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	9	10	11	12	$^{s}\Gamma_1$	Γ_2	Γ_3	Γ_4	Γ_5
14.1	$D^{i}_{2d}/C^{k}_{1}(D_4, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2†
14.2	$D^{i}_{2d}/C^{(x)j}_{2}/C^{1}_{1}(D_4, C'_2)_4$	j:	1	1	2	2	3	3	3	3	3	3	3	3	1	.	1	.	1†
14.3	$D^{i}_{2d}/C^{(xy)j}_{s}/C^{1}_{1}(D_4, C'_2)_4$	j:	3	4	3	4	1	2	2	2	3	4	3	4	1	.	.	1	1†
14.4	$D^{i}_{2d}/C^{(z)k}_{2}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	3	3	3	3	1	1	1	1	
14.5	$D^{i}_{2d}/C^{k}_{2v}(C_2, C_1)_2$	k:	11	13	11	13	1	3	8	10	20	21	18	19	1	.	.	1†	
14.6	$D^{i}_{2d}/D^{k}_{2}(C_2, C_1)_2$	k:	1	1	3	3	6	6	6	6	7	7	8	9	1	.	1†		
14.7	$D^{i}_{2d}/S^{k}_{4}(C_2, C_1)_2$	k:	1	1	1	1	1	1	1	1	2	2	2	2	1	1†			
14.8	$D^{i}_{2d}/D^{k}_{2d}(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	11	12	1†				

Table 15. *Colour groups with* $\hat{G} = D_{4h}$

No.	$D^{i}_{4h}/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$^{s}\Gamma^{+}_1$	Γ^{-}_1	Γ^{+}_2	Γ_2	Γ^{+}_3	Γ^{-}_3	Γ^{+}_4	Γ_4	Γ^{+}_5	Γ^{-}_5
15.1	$D^{i}_{4h}/C^{k}_{1}(D_{4h}, C_1)_{16}$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2
15.2*a*	$D^{i}_{4h}/C^{(x)j}_{2}/C^{1}_{1}(D_{4h}, C'_2)_8$	j:	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	3	3	3	3	1	1	.	.	1	1	.	.	1	1
15.2*b*	$D^{i}_{4h}/C^{(xy)j}_{2}/C^{1}_{1}(D_{4h}, C'_2)_8$	j:	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	1	1	.	.	.	.	1	1	1	1
15.3*a*	$D^{i}_{4h}/C^{(x)j}_{s}/C^{1}_{1}(D_{4h}, C'_2)_8$	j:	1	2	2	2	2	2	1	2	1	2	2	2	2	2	1	2	3	4	3	4	1	.	.	1	1	.	.	1	1	1
15.3*b*	$D^{i}_{4h}/C^{(xy)j}_{s}/C^{1}_{1}(D_{4h}, C'_2)_8$	j:	3	4	3	4	3	4	3	4	4	3	4	3	4	3	4	3	3	3	4	4	1	.	.	1	.	1	1	.	1	1
15.4	$D^{i}_{4h}/C^{k}_{s}(D_4, C_1)_8$	k:	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	3	3	4	4	1	.	1	.	1	.	1	.	.	2†
15.5*a*	$D^{i}_{4h}/C^{(x)j}_{2v}/C^{k}_{s}(D_4, C'_2)_4$	j:	1	4	6	10	2	7	7	9	1	4	6	10	2	7	7	9	20	22	22	21	1	.	.	.	1	.	.	.	.	1†
15.5*b*	$D^{i}_{4h}/C^{(xy)j}_{2v}/C^{k}_{s}(D_4, C'_2)_4$	j:	14	16	15	17	14	16	15	17	16	14	17	15	16	14	17	15	18	18	19	19	1	.	.	.	.	.	1	.	.	1†
15.6	$D^{i}_{4h}/C^{k}_{i}(D_4, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	.	1	.	1	.	1	.	2†	
15.7*a*	$D^{i}_{4h}/C^{(x)j}_{2h}/C^{1}_{i}(D_4, C'_2)_4$	j:	1	4	4	4	5	5	2	5	1	4	4	4	5	5	2	5	3	6	3	6	1	.	.	.	1	.	.	.	1†	
15.7*b*	$D^{i}_{4h}/C^{(xy)j}_{2h}/C^{1}_{i}(D_4, C'_2)_4$	j:	3	6	3	6	3	6	3	6	6	3	6	3	6	3	6	3	3	3	6	6	1	.	.	.	.	.	1	.	1†	
15.8	$D^{i}_{4h}/C^{k}_{2}(D_{2h}, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	3	3	3	1	1	1	1	1	1	1	1		
15.9	$D^{i}_{4h}/C^{k}_{2h}(D_2, C_1)_4$	k:	1	1	4	4	1	1	4	4	1	1	4	4	1	1	4	4	3	3	6	6	1	.	1	.	1	.	1	.		
15.10*a*	$D^{i}_{4h}/D^{k}_{2}(D_2, C_1)_4$	k:	1	1	1	1	3	3	3	3	1	1	1	1	3	3	3	3	8	8	9	9	1	1	.	.	1	1	.	.		
15.10*b*	$D^{i}_{4h}/D'^{k}_{2}(D_2, C_1)_4$	k:	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7	1	1	.	.	.	.	1	1		
15.11*a*	$D^{i}_{4h}/C^{k}_{2v}(D_2, C_1)_4$	k:	1	3	8	10	8	10	1	3	1	3	8	10	8	10	1	3	20	21	20	21	1	.	.	1	1	.	.	1		
15.11*b*	$D^{i}_{4h}/C'^{k}_{2v}(D_2, C_1)_4$	k:	11	13	11	13	11	13	11	13	13	11	13	11	13	11	13	11	18	18	19	19	1	.	.	1	.	1	1	.		
15.12	$D^{i}_{4h}/C^{k}_{4}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	3	3	3	3	3	3	3	3	5	5	6	6	1	1	1	1	.	.	.	.		
15.13	$D^{i}_{4h}/S^{k}_{4}(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	1	.	1	.	.	1	.	1		
15.14*a*	$D^{i}_{4h}/D'^{k}_{2d}(C_2, C_1)_2$	k:	5	6	7	8	7	8	5	6	5	6	7	8	7	8	5	6	9	10	9	10	1	.	.	.	.	.	.	1†		
15.15*a*	$D^{i}_{4h}/D'^{k}_{2h}(C_2, C_1)_2$	k:	19	20	21	22	19	20	21	22	20	19	22	21	20	19	22	21	23	23	24	24	1	.	.	.	.	.	1†			
15.14*b*	$D^{i}_{4h}/D^{k}_{2d}(C_2, C_1)_2$	k:	1	2	1	2	3	4	3	4	2	1	2	1	4	3	4	3	11	11	12	12	1	.	.	.	.	1†				
15.15*b*	$D^{i}_{4h}/D^{k}_{2h}(C_2, C_1)_2$	k:	1	3	4	2	9	12	13	10	1	3	4	2	9	12	13	10	25	26	28	27	1	.	.	.	1†					
15.16	$D^{i}_{4h}/C^{k}_{4v}(C_2, C_1)_2$	k:	1	5	2	6	2	6	1	5	7	3	8	4	8	4	7	3	9	10	11	12	1	.	.	1†						
15.17	$D^{i}_{4h}/C^{k}_{4h}(C_2, C_1)_2$	k:	1	1	3	3	1	1	3	3	2	2	4	4	2	2	4	4	5	5	6	6	1	.	1†							
15.18	$D^{i}_{4h}/D^{k}_{4}(C_2, C_1)_2$	k:	1	1	1	1	2	2	2	2	5	5	5	5	6	6	6	6	9	9	10	10	1	1†								
15.19	$D^{i}_{4h}/D^{k}_{4h}(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1†									

Table 16. *Colour groups with* $\hat{G} = C_3$

No.	$C_3^i/H'/H(A, A')_n$	i:	1	2	3	4	$^s\Gamma_1$	$(^s\Gamma_2$	$^s\Gamma_3)^h$
16.1	$C_3^i/C_1^k(C_3, C_1)_3$	k:	1	1	1	1	1	1†	1†
16.2	$C_3^i/C_3^k(C_1, C_1)_1$	k:	1	2	3	4	1†		

Table 17. *Colour groups with* $\hat{G} = C_{3i}$

No.	$C_{3i}^i/H'/H(A, A')_n$	i:	1	2	$^s\Gamma_1^+$	$(^s\Gamma_2^+$	$^s\Gamma_3^+)$	Γ_1^-	$(\Gamma_2^-$	$\Gamma_3^-)$
17.1	$C_{3i}^i/C_1^k(C_6, C_1)_6$	k:	1	1	1	1	1	1	1†	1†
17.2	$C_{3i}^i/C_i^k(C_3, C_1)_3$	k:	1	1	1	1†	1†	.		
17.3	$C_{3i}^i/C_3^k(C_2, C_1)_2$	k:	1	4	1	.	.	1†		
17.4	$C_{3i}^i/C_{3i}^k(C_1, C_1)_1$	k:	1	2	1†					

Table 18. *Colour groups with* $\hat{G} = D_3$

No.	$D_3^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	$^s\Gamma_1$	Γ_2	$^s\Gamma_3^h$
18.1	$D_3^i/C_1^k(D_3, C_1)_6$	k:	1	1	1	1	1	1	1	1	1	2†
18.2	$D_3^i/C_2^j/C_1^1(D_3, C_2)_3$	j:	3	3	3	3	3	3	3	1	.	1†
18.3	$D_3^i/C_3^k(C_2, C_1)_2$	k:	1	1	2	2	3	3	4	1	1†	
18.4	$D_3^i/D_3^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	1†		

Table 19. *Colour groups with* $\hat{G} = C_{3v}$

No.	$C_{3v}^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	$^s\Gamma_1$	Γ_2	$^s\Gamma_3$
19.1	$C_{3v}^i/C_1^k(D_3, C_1)_6$	k:	1	1	1	1	1	1	1	1	2†
19.2	$C_{3v}^i/C_s^j/C_1^1(D_3, C_2)_3$	j:	3	3	4	4	3	4	1	.	1†
19.3	$C_{3v}^i/C_3^k(C_2, C_1)_2$	k:	1	1	1	1	4	4	1	1†	
19.4	$C_{3v}^i/C_{3v}^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†		

Table 20. *Colour groups with* $\hat{G} = D_{3d}$

No.	$D_{3d}^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	$^s\Gamma_1^+$	Γ_2^+	Γ_1^-	Γ_2^-	$^s\Gamma_3^+$	Γ_3^-
20.1	$D_{3d}^i/C_1^k(D_6, C_1)_{12}$	k:	1	1	1	1	1	1	1	1	1	1	2	2†
20.2	$D_{3d}^i/C_2^j/C_1^1(D_6, C_2')_6$	j:	3	3	3	3	3	3	1	.	1	.	1	1†
20.3	$D_{3d}^i/C_s^j/C_1^1(D_6, C_2')_6$	j:	3	4	3	4	3	4	1	.	.	1	1	1†
20.4	$D_{3d}^i/C_i^k(D_3, C_1)_6$	k:	1	1	1	1	1	1	1	1	.	.	2†	
20.5	$D_{3d}^i/C_{2h}^j/C_i^1(D_3, C_2)_3$	j:	3	6	3	6	3	6	1	.	.	.	1†	
20.6	$D_{3d}^i/C_3^k(D_2, C_1)_4$	k:	1	1	1	1	4	4	1	1	1	1		
20.7	$D_{3d}^i/C_{3v}^k(C_2, C_1)_2$	k:	2	4	1	3	5	6	1	.	.	1†		
20.8	$D_{3d}^i/D_3^k(C_2, C_1)_2$	k:	1	1	2	2	7	7	1	.	1†			
20.9	$D_{3d}^i/C_{3i}^k(C_2, C_1)_2$	k:	1	1	1	1	2	2	1	1†				
20.10	$D_{3d}^i/D_{3d}^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†					

Table 21. *Colour groups with* $\hat{G} = C_6$

No.	$C_6^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	$^s\Gamma_1$	$(^s\Gamma_2$	$^s\Gamma_3)^h$	Γ_4	$(\Gamma_5$	$\Gamma_6)^h$
21.1	$C_6^i/C_1^k(C_6, C_1)_6$	k:	1	1	1	1	1	1	1	1	1	1	1†	1†
21.2	$C_6^i/C_2^k(C_3, C_1)_3$	k:	1	2	2	1	1	2	1	1†	1†	.		
21.3	$C_6^i/C_3^k(C_2, C_1)_2$	k:	1	2	3	3	2	1	1	.	.	1†		
21.4	$C_6^i/C_6^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†					

Table 22. *Colour groups with* $\hat{G} = C_{3h}$

No.	$C_{3h}^i/H'/H(A, A')_n$	i:	1	$^s\Gamma_1$	$(^s\Gamma_2$	$^s\Gamma_3)$	Γ_4	$(\Gamma_5$	$\Gamma_6)$
22.1	$C_{3h}^i/C_1^k(C_6, C_1)_6$	k:	1	1	1	1	1	1†	1†
22.2	$C_{3h}^i/C_s^k(C_3, C_1)_3$	k:	1	1	1†	1†	.		
22.3	$C_{3h}^i/C_3^k(C_2, C_1)_2$	k:	1	1	.	.	1†		
22.4	$C_{3h}^i/C_{3h}^k(C_1, C_1)_1$	k:	1	1†					

Table 23. *Colour groups with* $\hat{G} = C_{6h}$

No.	$C_{6h}^i/H'/H(A, A')_n$	i: 1 2	${}^s\Gamma_1^+$	Γ_4^+	Γ_1^-	Γ_4^-	$({}^s\Gamma_2^+$	${}^s\Gamma_3^+)$	$(\Gamma_5^+$	$\Gamma_6^+)$	$(\Gamma_2^-$	$\Gamma_3^-)$	$(\Gamma_5^-$	$\Gamma_6^-)$
23.1	$C_{6h}^i/C_1^k(C_{6h}, C_1)_{12}$	k: 1 1	1	1	1	1	1	1	1	1	1	1	1	1
23.2	$C_{6h}^i/C_s^k(C_6, C_1)_6$	k: 1 1	1	.	.	1	1	1	.	.	.	.	1†	1†
23.3	$C_{6h}^i/C_2^k(C_6, C_1)_6$	k: 1 2	1	.	1	.	1	1	.	.	1†	1†		
23.4	$C_{6h}^i/C_i^k(C_6, C_1)_6$	k: 1 1	1	1	.	.	1	1	1†	1†				
23.5	$C_{6h}^i/C_{2h}^k(C_3, C_1)_3$	k: 1 2	1	.	.	.	1†	1†						
23.6	$C_{6h}^i/C_3^k(D_2, C_1)_4$	k: 1 1	1	1	1	1								
23.7	$C_{6h}^i/C_{3h}^k(C_2, C_1)_2$	k: 1 1	1	.	.	1†								
23.8	$C_{6h}^i/C_6^k(C_2, C_1)_2$	k: 1 6	1	.	1†									
23.9	$C_{6h}^i/C_{3i}^k(C_2, C_1)_2$	k: 1 1	1	1†										
23.10	$C_{6h}^i/C_{6h}^k(C_1, C_1)_1$	k: 1 2	1†											

Table 24. *Colour groups with* $\hat{G} = D_6$

No.	$D_6^i/H'/H(A, A')_n$	i: 1 2 3 4 5 6	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4	${}^s\Gamma_6^h$	Γ_5^h
24.1	$D_6^i/C_1^k(D_6, C_1)_{12}$	k: 1 1 1 1 1 1	1	1	1	1	2†	2†
24.2*a*	$D_6^i/C_2'^j/C_1^1(D_6, C_2')_6$	j: 3 3 3 3 3 3	1	.	1	.	1	1†
24.2*b*	$D_6^i/C_2''^j/C_1^1(D_6, C_2')_6$	j: 3 3 3 3 3 3	1	.	.	1	1	1†
24.3	$D_6^i/C_2^k(D_3, C_1)_6$	k: 1 2 2 1 1 2	1	1	.	.	2†	
24.4	$D_6^i/D_2^j/C_2^k(D_3, C_2)_3$	j: 6 5 5 6 6 5	1	.	.	.	1†	
24.5	$D_6^i/C_3^k(D_2, C_1)_4$	k: 1 2 3 3 2 1	1	1	1	1		
24.6*a*	$D_6^i/D_3^k(C_2, C_1)_2$	k: 2 4 6 6 4 2	1	.	.	1†		
24.6*b*	$D_6^i/D_3'^k(C_2, C_1)_2$	k: 1 3 5 5 3 1	1	.	1†			
24.7	$D_6^i/C_6^k(C_2, C_1)_2$	k: 1 2 3 4 5 6	1	1†				
24.8	$D_6^i/D_6^k(C_1, C_1)_1$	k: 1 2 3 4 5 6	1†					

Table 25. *Colour groups with* $\hat{G} = C_{6v}$

No.	$C_{6v}^i/H'/H(A, A')_n$	i: 1 2 3 4	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4	${}^s\Gamma_6$	Γ_5
25.1	$C_{6v}^i/C_1^k(D_6, C_1)_{12}$	k: 1 1 1 1	1	1	1	1	2†	2†
25.2*a*	$C_{6v}^i/C_s'^j/C_1^1(D_6, C_2')_6$	j: 3 4 3 4	1	.	1	.	1	1†
25.2*b*	$C_{6v}^i/C_s''^j/C_1^1(D_6, C_2')_6$	j: 3 4 4 3	1	.	.	1	1	1†
25.3	$C_{6v}^i/C_2^k(D_3, C_1)_6$	k: 1 1 2 2	1	1	.	.	2†	
25.4	$C_{6v}^i/C_{2v}^j/C_2^k(D_3, C_2)_3$	j: 11 13 12 12	1	.	.	.	1†	
25.5	$C_{6v}^i/C_3^k(D_2, C_1)_4$	k: 1 1 1 1	1	1	1	1		
25.6*a*	$C_{6v}^i/C_{3v}'^k(C_2, C_1)_2$	k: 1 3 3 1	1	.	.	1†		
25.6*b*	$C_{6v}^i/C_{3v}^k(C_2, C_1)_2$	k: 2 4 2 4	1	.	1†			
25.7	$C_{6v}^i/C_6^k(C_2, C_1)_2$	k: 1 1 6 6	1	1†				
25.8	$C_{6v}^i/C_{6v}^k(C_1, C_1)_1$	k: 1 2 3 4	1†					

Table 26. *Colour groups with* $\hat{G} = D_{3h}$

No.	$D^i_{3h}/H'/H(A, A')_n$	i:	1	2	3	4	${}^s\Gamma_1$	Γ_2	Γ_3	Γ_4	${}^s\Gamma_6$	Γ_5
26.1	$D^i_{3h}/C^k_1(D_6, C_1)_{12}$	k:	1	1	1	1	1	1	1	1	2†	2†
26.2	$D^i_{3h}/C^j_2/C^1_1(D_6, C'_2)_6$	j:	3	3	3	3	1	.	1	.	1	1†
26.3	$D^i_{3h}/C'^j_s/C^1_1(D_6, C'_2)_6$	j:	3	4	3	4	1	.	.	1	1	1†
26.4	$D^i_{3h}/C^k_s(D_3, C_1)_6$	k:	1	1	1	1	1	1	.	.	2†	
26.5	$D^i_{3h}/C^j_{2v}/C^k_s(D_3, C_2)_3$	j:	14	16	14	16	1	.	.	.	1†	
26.6	$D^i_{3h}/C^k_3(D_2, C_1)_4$	k:	1	1	1	1	1	1	1	1		
26.7	$D^i_{3h}/C^k_{3v}(C_2, C_1)_2$	k:	1	3	2	4	1	.	.	1†		
26.8	$D^i_{3h}/D^k_3(C_2, C_1)_2$	k:	1	1	2	2	1	.	1†			
26.9	$D^i_{3h}/C^k_{3h}(C_2, C_1)_2$	k:	1	1	1	1	1	1†				
26.10	$D^i_{3h}/D^k_{3h}(C_1, C_1)_1$	k:	1	2	3	4	1†					

bears in mind the following. All the colour-group symbols with the same H (*i.e.* the same index k) are collected in separated blocks, in the first row of which the 'reduced' symbol $G/H(A, C_1)_n$ and the index k of the group H are given. The index j of H' can be found in the row of the corresponding colour-group symbol $G/H'/H(A, A')_n$.

For example, the ten colour space groups $O^i_h/D^j_4/D^k_2(D_6, C'_2)_6$ in Table 32 can be derived from the point group $O_h/D_4/D_2(D_6, C'_2)_6$ by using the following index correspondence:

$$\begin{array}{llcccccccccc} O^i_h, & i= & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ D^j_4, & j= & 1 & 1 & 5 & 5 & 9 & 9 & 10 & 10 & 9 & 10 \\ D^k_2, & k= & 1 & 1 & 1 & 1 & 7 & 7 & 7 & 7 & 8 & 9. \end{array} \quad (25)$$

In the list the running number of each $G^{(P)} \equiv T^{(1)}\hat{G}^{(P)}$ can be presented as Ni, where N is the number of its colour point group $\hat{G}^{(P)}$, used in the tables of Litvin, Kotzev & Birman (1982) and given in the 'No.' column, and i is the upper index in the Schoenflies symbol of G. Running numbers from 32.25.1 to 32.25.10, respectively, should be assigned to the ten groups in our example (25).

In Table 33 the numbers of the isogonal space groups G, the isostructural colour point groups $\hat{G}^{(P)}$ and the colour space groups $T^{(1)}\hat{G}^{(P)}$ engendered by them are given for each of the 32 point groups $\hat{G}$. Owing to the equivalence definition used the number of $G^{(P)}$ is equal to the product of the isogonal space-group number and that of the isostructural colour point groups $\hat{G}^{(P)}$. Generally this is not true for any other equivalence definition.

In Table 34 the 2571 groups $T^{(1)}\hat{G}^{(P)}$ are classified in the 45 chromomorphic classes $(A, A')_n$. In the columns $\hat{N}_{\mathscr{A}^+}$ and $\hat{N}_{\hat{G}}$ the colour-point-group numbers are given in the sense of the equivalence definition (23) and (24), respectively, for each $(A, A')_n$. The number of the space groups $T^{(1)}\hat{G}^{(P)}$ derived using our equivalence definition (24) is given in the N_G column. Lists of colour space groups belonging only to some chromomorphic classes have been published by other authors. The number of the colour space groups derived by them are given in the 'Other lists' column. There is no discrepancy between their and our tables, if one takes into account the difference in the equivalence definition. Koptsik & Kuzhukeev (1972) have published the full list of colour space groups belonging to the chromomorphic class $(D_3, C_1)_6$ and only the number of those belonging to the classes $(D_2, C_1)_4$ and $(C_n, C_1)_n$, $n = 3, 4, 6$.

Rama Mohana Rao & Kondala Rao (1983) have used a method for constructing some colour cubic space groups, which was originally proposed by Shubnikov & Koptsik (1974). But the list of the derived colour groups is incomplete and contains some errors. For example, the group $P2_13^{(6)}$ in their tables should not exist; it is not included in the tables of Zamorzaev *et al.* (1978) either. Neither can the groups derived as its extensions exist.

In Tables 1 to 32 the multiplicity $(D^{H'}_G | D^j_G)$ with which the irreducible representations D^j_G of G appear in the associated permutational representations $D^{H'}_G = \sum_j (D^{H'}_G | D^j_G) D^j_G$ is tabulated.

The permutational representation $D^{H'}_G$ of the colour space group with colour-preserving lattice $T^{(1)}\hat{G}^{(P)}$ is engendered by the point-group permutational representation $D^{\hat{H}'}_{\hat{G}}$:

$$D^{H'}_G = \varphi^{-1}[D^{\hat{H}'}_{\hat{G}}] = D^{\hat{H}'}_{\hat{G}} \uparrow\uparrow G. \quad (26)$$

The multiplicity of the irreducible representation $D^j_{\hat{G}} \equiv \Gamma_j$ of $\hat{G}$ in $D^{\hat{H}'}_{\hat{G}}$ is equal to the multiplicity of $k = 0$ irreducible representation $D^{(0)j}_G$ of G, engendered by Γ_j, *i.e.*

$$(D^{H'}_G | D^{(0)j}_G) = (D^{H'}_G | (\Gamma_j \uparrow\uparrow G)) = (D^{\hat{H}'}_{\hat{G}} | \Gamma_j). \quad (27)$$

Therefore, the notation of space-group-irreducible representations $D^{(0)j}_G = (\Gamma_j \uparrow\uparrow G)$ is given for short in

Table 27. *Colour groups with $\hat{G} = D_{6h}$*

No.	$D^i_{6h}/H'/H(A, A')_n$	i:	1	2	3	4	$^s\Gamma_1^+$	Γ_1^-	Γ_2^+	Γ_2^-	Γ_3^+	Γ_3^-	Γ_4^+	Γ_4^-	$^s\Gamma_6^+$	Γ_6^-	Γ_5^+	Γ_5^-
27.1	$D^i_{6h}/C_1^k(D_{6h}, C_1)_{24}$	k:	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2
27.2*a*	$D^i_{6h}/C_2^{\prime j}/C_1^1(D_{6h}, C_2')_{12}$	j:	3	3	3	3	1	1	.	.	.	.	1	1	1	1	1	1
27.2*b*	$D^i_{6h}/C_2^{\prime\prime j}/C_1^1(D_{6h}, C_2')_{12}$	j:	3	3	3	3	1	1	.	.	1	1	.	.	1	1	1	1
27.3*a*	$D^i_{6h}/C_s^{\prime j}/C_1^1(D_{6h}, C_2')_{12}$	j:	3	4	4	3	1	.	.	1	.	1	1	.	1	1	1	1
27.3*b*	$D^i_{6h}/C_s^{\prime\prime j}/C_1^1(D_{6h}, C_2')_{12}$	j:	3	4	3	4	1	.	.	1	1	.	.	1	1	1	1	1
27.4	$D^i_{6h}/C_s^k(D_6, C_1)_{12}$	k:	1	1	1	1	1	.	1	.	.	1	.	1	2	.	.	2†
27.5*a*	$D^i_{6h}/C_{2v}^{\prime j}/C_s^k(D_6, C_2')_6$	j:	14	16	14	16	1	.	.	.	.	.	.	1	1	.	.	1†
27.5*b*	$D^i_{6h}/C_{2v}^{\prime\prime j}/C_s^k(D_6, C_2')_6$	j:	14	16	16	14	1	.	.	.	.	1	.	.	1	.	.	1†
27.6	$D^i_{6h}/C_i^k(D_6, C_1)_{12}$	k:	1	1	1	1	1	.	1	.	1	.	1	.	2	.	2†	
27.7*a*	$D^i_{6h}/C_{2h}^{\prime j}/C_i^k(D_6, C_2')_6$	j:	3	6	6	3	1	.	.	.	.	.	1	.	1	.	1†	
27.7*b*	$D^i_{6h}/C_{2h}^{\prime\prime j}/C_i^k(D_6, C_2')_6$	j:	3	6	3	6	1	.	.	.	1	.	.	.	1	.	1†	
27.8	$D^i_{6h}/C_2^k(D_6, C_1)_{12}$	k:	1	1	2	2	1	1	1	1	.	.	.	.	2	2†		
27.9	$D^i_{6h}/D_2^j/C_2^k(D_6, C_2')_6$	j:	6	6	5	5	1	1	.	.	.	.	.	.	1	1†		
27.10	$D^i_{6h}/C_{2v}^j/C_2^k(D_6, C_2')_6$	j:	11	13	12	12	1	.	.	1	.	.	.	.	1	1†		
27.11	$D^i_{6h}/C_{2h}^k(D_3, C_1)_6$	k:	1	1	2	2	1	.	1	.	.	.	.	.	2†			
27.12	$D^i_{6h}/D_{2h}^j/C_{2h}^k(D_3, C_2)_3$	j:	19	20	17	17	1	.	.	.	.	.	.	.	1†			
27.13	$D^i_{6h}/C_3^k(D_{2h}, C_1)_8$	k:	1	1	1	1	1	1	1	1	1	1	1	1				
27.14	$D^i_{6h}/C_{3i}^k(D_2, C_1)_4$	k:	1	1	1	1	1	.	1	.	1	.	1	.				
27.15	$D^i_{6h}/C_6^k(D_2, C_1)_4$	k:	1	1	6	6	1	1	1	1	.	.	.	.				
27.16	$D^i_{6h}/C_{3h}^k(D_2, C_1)_4$	k:	1	1	1	1	1	.	1	.	.	1	.	1				
27.17*a*	$D^i_{6h}/D_3^k(D_2, C_1)_4$	k:	2	2	2	2	1	1	.	.	.	.	1	1				
27.17*b*	$D^i_{6h}/D_3^{\prime k}(D_2, C_1)_4$	k:	1	1	1	1	1	1	.	.	1	1	.	.				
27.18*a*	$D^i_{6h}/C_{3v}^k(D_2, C_1)_4$	k:	1	3	3	1	1	.	.	1	.	1	1	.				
27.18*b*	$D^i_{6h}/C_{3v}^{\prime k}(D_2, C_1)_4$	k:	2	4	2	4	1	.	.	1	1	.	.	1				
27.19*a*	$D^i_{6h}/D_{3h}^{\prime k}(C_2, C_1)_2$	k:	3	4	3	4	1	.	.	.	.	.	.	1†				
27.20*a*	$D^i_{6h}/D_{3d}^k(C_2, C_1)_2$	k:	3	4	4	3	1	.	.	.	.	.	1†					
27.19*b*	$D^i_{6h}/D_{3h}^k(C_2, C_1)_2$	k:	1	2	2	1	1	.	.	.	.	1†						
27.20*b*	$D^i_{6h}/D_{3d}^{\prime k}(C_2, C_1)_2$	k:	1	2	1	2	1	.	.	.	1†							
27.21	$D^i_{6h}/C_{6v}^k(C_2, C_1)_2$	k:	1	2	3	4	1	.	.	1†								
27.22	$D^i_{6h}/C_{6h}^k(C_2, C_1)_2$	k:	1	1	2	2	1	.	1†									
27.23	$D^i_{6h}/D_6^k(C_2, C_1)_2$	k:	1	1	6	6	1	1†										
27.24	$D^i_{6h}/D_{6h}^k(C_1, C_1)_1$	k:	1	2	3	4	1†											

Table 28. *Colour groups with $\hat{G} = T$*

No.	$T^i/H'/H(A, A')_n$	i:	1	2	3	4	5	$^s\Gamma_1$	($^s\Gamma_2$	$^s\Gamma_3$)	$^s\Gamma_4^h$
28.1	$T^i/C_1^k(T, C_1)_{12}$	k:	1	1	1	1	1	1	1	1	3†
28.2	$T^i/C_2^j/C_1^1(T, C_2)_6$	j:	1	3	3	2	3	1	1	1	1†
28.3	$T^i/C_3^j/C_1^1(T, C_3)_4$	j:	4	4	4	4	4	1	.	.	1†
28.4	$T^i/D_2^k(C_3, C_1)_3$	k:	1	7	8	4	9	1	1†	1†	
28.5	$T^i/T^k(C_1, C_1)_1$	k:	1	2	3	4	5	1†			

the first lines of the tables by the symbols of the related point-group irreducible representations Γ_j, using the symbols and enumeration of Koster, Dimmok, Wheeler & Statz (1963).

In our example (25) all the ten representations $D_{O_h^i}^{D_4^j}$ are engendered by the same permutational representation of the point group $D_{O_h}^{D_4} = D_{D_6}^{C_2^j} \uparrow\uparrow O_h = \Gamma_1^+ + \Gamma_1^- + \Gamma_3^+ + \Gamma_3^-$:

$$\begin{aligned} D_{O_h^i}^{D_4^j} &= D_{O_h}^{D_4} \uparrow\uparrow O_h^i \\ &= \Gamma_1^+ \uparrow\uparrow O_h^i + \Gamma_1^- \uparrow\uparrow O_h^i + \Gamma_3^+ \uparrow\uparrow O_h^i + \Gamma_3^- \uparrow\uparrow O_h^i \\ &= D^{(0)1+} \quad + D^{(0)1-} \quad + D^{(0)3+} \quad + D^{(0)3-}. \end{aligned} \qquad (28)$$

In our next paper (Kotzev & Alexandrova, 1988) the colour space groups with colour preserving translations and the corresponding tables are applied in the analysis of equi-translational phase transitions in

Table 29. *Colour groups with $\hat{G} = T_h$*

No.	$T_h^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	$^s\Gamma_1^+$	$(^s\Gamma_2^+$	$^s\Gamma_3^+)$	$^s\Gamma_4^+$	Γ_1^-	$(\Gamma_2^-$	$\Gamma_3^-)$	Γ_4^-
29.1	$T_h^i/C_1^k(T_h, C_1)_{24}$	k:	1	1	1	1	1	1	1	1	1	1	3	1	1	1	3†
29.2	$T_h^i/C_s^j/C_1^1(T_h, C_s)_{12}$	j:	1	2	3	4	3	2	4	1	1	1	1	.	.	.	2†
29.3	$T_h^i/C_2^j/C_1^1(T_h, C_2)_{12}$	j:	1	1	3	3	3	2	3	1	1	1	1	1	1	1	1
29.4	$T_h^i/C_3^j/C_1^1(T_h, C_3)_8$	j:	4	4	4	4	4	4	4	1	.	.	1	1	.	.	1†
29.5	$T_h^i/C_{2v}^j/C_1^1(T_h, C_{2v})_6$	j:	1	10	18	19	20	5	21	1	1	1	.	.	.	.	1†
29.6	$T_h^i/D_2^k(C_6, C_1)_6$	k:	1	1	7	7	8	4	9	1	1	1	.	1	1†	1†	
29.7	$T_h^i/T^k(C_2, C_1)_2$	k:	1	1	2	2	3	4	5	1	.	.	.	1†			
29.8	$T_h^i/C_i^k(T, C_1)_{12}$	k:	1	1	1	1	1	1	1	1	1	1	3†				
29.9	$T_h^i/C_{2h}^j/C_i^1(T, C_2)_6$	j:	1	4	3	6	3	5	6	1	1	1	1†				
29.10	$T_h^i/C_{3i}^j/C_i^1(T, C_3)_4$	j:	2	2	2	2	2	2	2	1	.	.	1†				
29.11	$T_h^i/D_{2h}^k(C_3, C_1)_3$	k:	1	2	23	24	25	15	27	1	1†	1†					
29.12	$T_h^i/T_h^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	1†							

Table 30. *Colour groups with $\hat{G} = O$*

No.	$O^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	7	8	$^s\Gamma_1$	Γ_2	$^s\Gamma_3$	Γ_4^h	$^s\Gamma_5^h$
30.1	$O^i/C_1^k(O, C_1)_{24}$	k:	1	1	1	1	1	1	1	1	1	1	2	3†	3†
30.2	$O^i/C_2^j/C_1^1(O, C_2)_{12}$	j:	1	1	3	3	3	2	2	3	1	1	2	1	1
30.3	$O^i/C_2'^j/C_1^1(O, C_2')_{12}$	j:	3	3	3	3	3	3	3	3	1	.	1	1†	2†
30.4	$O^i/C_3^j/C_1^1(O, C_3)_8$	j:	4	4	4	4	4	4	4	4	1	1	.	1†	1
30.5	$O^i/D_2'^j/C_1^1(O, D_2')_6$	j:	6	6	8	9	7	5	5	7	1	.	1	.	1†
30.6	$O^i/C_4^j/C_1^1(O, C_4)_6$	j:	1	3	5	6	5	4	2	6	1	.	1	1†	.
30.7	$O^i/D_3^j/C_1^1(O, D_3)_4$	j:	7	7	7	7	7	7	7	7	1	.	.	.	1†
30.8	$O^i/D_2^k(D_3, C_1)_6$	k:	1	1	7	7	8	4	4	9	1	1	2†		
30.9	$O^i/D_4^j/D_2^k(D_3, C_2)_3$	j:	1	5	9	10	9	8	4	10	1	.	1†		
30.10	$O^i/T^k(C_2, C_1)_2$	k:	1	1	2	2	3	4	4	5	1	1†			
30.11	$O^i/O^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	7	8	1†				

Table 31. *Colour groups with $\hat{G} = T_d$*

No.	$T_d^i/H'/H(A, A')_n$	i:	1	2	3	4	5	6	$^s\Gamma_1$	Γ_2	$^s\Gamma_3$	Γ_4	$^s\Gamma_5$
31.1	$T_d^i/C_1^k(O, C_1)_{24}$	k:	1	1	1	1	1	1	1	1	2	3†	3†
31.2	$T_d^i/C_2^j/C_1^1(O, C_2)_{12}$	j:	1	3	3	1	3	3	1	1	2	1	1
31.3	$T_d^i/C_s^j/C_1^1(O, C_2')_{12}$	j:	3	3	3	4	4	4	1	.	1	1†	2†
31.4	$T_d^i/C_3^j/C_1^1(O, C_3)_8$	j:	4	4	4	4	4	4	1	1	.	1†	1
31.5	$T_d^i/C_{2v}^j/C_1^1(O, D_2')_6$	j:	11	20	18	13	21	19	1	.	1	.	1†
31.6	$T_d^i/S_4^j/C_1^1(O, C_4)_6$	j:	1	2	2	1	2	2	1	.	1	1†	.
31.7	$T_d^i/C_{3v}^j/C_1^1(O, D_3)_4$	j:	5	5	5	6	6	6	1	.	.	.	1†
31.8	$T_d^i/D_2^k(D_3, C_1)_6$	k:	1	7	8	1	7	9	1	1	2†		
31.9	$T_d^i/D_{2d}^j/D_2^k(D_3, C_2)_3$	j:	1	9	11	2	10	12	1	.	1†		
31.10	$T_d^i/T^k(C_2, C_1)_2$	k:	1	2	3	1	2	5	1	1†			
31.11	$T_d^i/T_d^k(C_1, C_1)_1$	k:	1	2	3	4	5	6	1†				

Table 32. *Colour groups with* $\hat{G} = O_h$

No.	$O_h^i/H'/H(A,A')_n$	i:	1	2	3	4	5	6	7	8	9	10	${}^s\Gamma_1^+$	Γ_2^+	${}^s\Gamma_3^+$	Γ_4^+	${}^s\Gamma_5^+$	Γ_1^-	Γ_2^-	Γ_3^-	Γ_4^-	Γ_5^-
32.1	$O_h^i/C_1^k(O_h,C_1)_{48}$	k:	1	1	1	1	1	1	1	1	1	1	1	1	2	3	3	1	1	2	3†	3†
32.2	$O_h^i/C_2^j/C_1^1(O_h,C_2)_{24}$	j:	1	1	1	1	3	3	3	3	3	3	1	1	2	1	1	1	1	2	1	1
32.3	$O_h^i/C_s^j/C_1^1(O_h,C_s)_{24}$	j:	1	2	1	2	3	3	4	4	3	4	1	1	2	1	1	.	.	.	2†	2†
32.4	$O_h^i/C_2'^j/C_1^1(O_h,C_2')_{24}$	j:	3	3	3	3	3	3	3	3	3	3	1	.	1	1	2	1	.	1	1	2†
32.5	$O_h^i/C_s'^j/C_1^1(O_h,C_s')_{24}$	j:	3	4	4	3	3	4	3	4	3	4	1	.	1	1	2	.	1	1	2†	1
32.6	$O_h^i/C_3^j/C_1^1(O_h,C_3)_{16}$	j:	4	4	4	4	4	4	4	4	4	4	1	1	.	1	1	1	1	.	1	1
32.7	$O_h^i/C_4^j/C_1^1(O_h,C_4)_{12}$	j:	1	1	3	3	5	5	6	6	5	6	1	.	1	1	.	1	.	1	1	.
32.8	$O_h^i/S_4^j/C_1^1(O_h,C_4)_{12}$	j:	1	1	1	1	2	2	2	2	2	2	1	.	1	1	.	.	1	1	.	1
32.9	$O_h^i/C_{2v}^j/C_1^1(O_h,C_{2v})_{12}$	j:	1	10	1	10	18	18	19	19	20	21	1	1	2	.	.	.	.	.	1	1
32.10	$O_h^i/D_2'^j/C_1^1(O_h,D_2')_{12}$	j:	6	6	6	6	8	8	9	9	7	7	1	.	1	.	1	1	.	1	.	1
32.11	$O_h^i/C_{2v}'^j/C_1^1(O_h,D_2')_{12}$	j:	11	13	13	11	20	21	20	21	18	19	1	.	1	.	1	.	1	1	1	.
32.12	$O_h^i/C_{2v}''^j/C_1^1(O_h,C_{2v}'')_{12}$	j:	14	17	16	15	20	22	22	21	18	19	1	.	1	.	1	.	.	.	1†	1†
32.13	$O_h^i/D_3^j/C_1^1(O_h,D_3)_8$	j:	7	7	7	7	7	7	7	7	7	7	1	.	.	.	1	1	.	.	.	1†
32.14	$O_h^i/C_{3v}^j/C_1^1(O_h,D_3)_8$	j:	5	6	6	5	5	6	5	6	5	6	1	.	.	.	1	.	1	.	1†	.
32.15	$O_h^i/C_{4v}^j/C_1^1(O_h,C_{4v})_6$	j:	1	6	7	4	9	10	11	12	9	12	1	.	1	.	.	.	.	.	1†	.
32.16	$O_h^i/D_{2d}'^j/C_1^1(O_h,C_{4v})_6$	j:	5	8	5	8	11	11	12	12	9	10	1	.	1	.	.	.	.	.	.	1†
32.17	$O_h^i/C_i^k(O,C_1)_{24}$	k:	1	1	1	1	1	1	1	1	1	1	1	1	2	3†	3†	.	.	.		
32.18	$O_h^i/C_{2h}^j/C_i^1(O,C_2)_{12}$	j:	1	4	1	4	3	3	6	6	3	6	1	1	2	1	1	.	.	.		
32.19	$O_h^i/C_{2h}'^j/C_i^1(O,C_2')_{12}$	j:	3	6	6	3	3	6	3	6	3	6	1	.	1	1†	2†	.	.	.		
32.20	$O_h^i/C_{3i}^j/C_i^1(O,C_3)_8$	j:	2	2	2	2	2	2	2	2	2	2	1	1	.	1†	1	.	.	.		
32.21	$O_h^i/C_{4h}^j/C_i^1(O,C_4)_6$	j:	1	3	2	4	5	5	6	6	5	6	1	.	1	1†	.	.	.	.		
32.22	$O_h^i/D_{2h}'^j/C_i^1(O,D_2')_6$	j:	19	22	20	21	25	26	28	27	23	24	1	.	1	.	1†	.	.	.		
32.23	$O_h^i/D_{3d}^j/C_i^1(O,D_3)_4$	j:	5	6	6	5	5	6	5	6	5	6	1	.	.	.	1†	.	.	.		
32.24	$O_h^i/D_2^k(D_6,C_1)_{12}$	k:	1	1	1	1	7	7	7	7	8	9	1	1	2	.	.	1	1	2†		
32.25	$O_h^i/D_4^j/D_2^k(D_6,C_2')_6$	j:	1	1	5	5	9	9	10	10	9	10	1	.	1	.	.	1	.	1†		
32.26	$O_h^i/D_{2d}^j/D_2^k(D_6,C_2')_6$	j:	1	2	2	1	9	10	9	10	11	12	1	.	1	.	.	.	1	1†		
32.27	$O_h^i/D_{2h}^k(D_3,C_1)_6$	k:	1	2	1	2	23	23	24	24	25	27	1	1	2†	.	.	.	.			
32.28	$O_h^i/D_{4h}^j/D_{2h}^k(D_3,C_2)_3$	j:	1	4	9	12	17	18	19	20	17	20	1	.	1†	.	.	.	.			
32.29	$O_h^i/T^k(D_2,C_1)_4$	k:	1	1	1	1	2	2	2	2	3	5	1	1	.	.	.	1	1			
32.30	$O_h^i/T_d^k(C_2,C_1)_2$	k:	1	4	4	1	2	5	2	5	3	6	1	.	.	.	.	.	1†			
32.31	$O_h^i/O^k(C_2,C_1)_2$	k:	1	1	2	2	3	3	4	4	5	8	1	.	.	.	.	1†				
32.32	$O_h^i/T_h^k(C_2,C_1)_2$	k:	1	2	1	2	3	3	4	4	5	7	1	1†								
32.33	$O_h^i/O_h^k(C_1,C_1)_1$	k:	1	2	3	4	5	6	7	8	9	10	1†									

Table 33. *Crystallographic classification of* $G^{(P)} = T^{(1)}\hat{G}^{(P)}$

No.	$\hat{G}$	G	$\hat{G}^{(P)}$	$G^{(P)}$	No.	$\hat{G}$	G	$\hat{G}^{(P)}$	$G^{(P)}$
1	C_1	1	1	1	17	C_{3i}	2	4	8
2	C_i	1	2	2	18	D_3	7	4	28
3	C_2	3	2	6	19	C_{3v}	6	4	24
4	C_s	4	2	8	20	D_{3d}	6	10	60
5	C_{2h}	6	5	30	21	C_6	6	4	24
6	D_2	9	5	45	22	C_{3h}	1	4	4
7	C_{2v}	22	5	110	23	C_{6h}	2	10	20
8	D_{2h}	28	16	448	24	D_6	6	10	60
9	C_4	6	3	18	25	C_{6v}	4	10	40
10	S_4	2	3	6	26	D_{3h}	4	10	40
11	C_{4h}	6	8	48	27	D_{6h}	4	32	128
12	D_4	10	8	80	28	T	5	5	25
13	C_{4v}	12	8	96	29	T_h	7	12	84
14	D_{2d}	12	8	96	30	O	8	11	88
15	D_{4h}	20	27	540	31	T_d	6	11	66
16	C_3	4	2	8	32	O_h	10	33	330
					Total		230	279	2571

two main directions: the classification of the phase transitions and the application of the known group-theoretical criteria required in the Landau theory in terms of colour groups (Birman, 1978; Litvin, Kotzev & Birman, 1982).

The irreducible representations Γ_j listed in Tables 1 to 32 with superscript 's', ${}^s\Gamma_j$, do not satisfy the Landau stability criterion. The physically irreducible representations Γ_j^h or $(\Gamma_j + \Gamma_j^*)$ with a superscript 'h' do not satisfy the Lifshitz homogeneity criterion. The multiplicity number $(D_G^{H'} | D_G^{(0)j})$ is marked with a dagger if for the triad $(G, H', D_G^{(0)j})$ the Birman chain-subduction criterion is satisfied [G and H' are in the symbol $G/H'/H(A,A')_n$, and $D_G^{(0)j} \leftrightarrow \Gamma_j$ is in the head of the corresponding column].

For 1587 of all 2571 groups $T^{(1)}\hat{G}^{(P)}$ the Birman chain-subduction criterion is satisfied. This number,

Table 34. *Chromomorphic classification of the* $G^{(P)} = T^{(1)}\hat{G}^{(P)}$

No.	$(A, A')_n$	$\hat{N}_{\mathscr{A}^+}$	$\hat{N}_{\hat{G}}$	N_G	Other lists
1	$(C_1, C_1)_1$	32	32	230	230
2	$(C_2, C_1)_2$	58	73	750	674*
3	$(C_3, C_1)_3$	7	7	27	27†‡§
4	$(C_4, C_1)_4$	4	4	20	20†§
5	$(C_6, C_1)_6$	7	7	22	22†§
6	$(C_{4h}, C_1)_8$	1	1	6	
7	$(C_{6h}, C_1)_{12}$	1	1	2	
8	$(D_2, C_1)_4$	26	34	473	421§
9	$(D_{2h}, C_1)_8$	3	3	52	
10	$(D_3, C_1)_6$	10	10	61	61§
11	$(D_3, C_2)_3$	10	10	61	61‡¶
12	$(D_4, C_1)_8$	5	5	74	
13	$(D_4, C'_2)_4$	6	10	148	148¶
14	$(D_{4h}, C_1)_{16}$	1	1	20	
15	$(D_{4h}, C'_2)_8$	2	4	80	
16	$(D_6, C_1)_{12}$	8	8	42	
17	$(D_6, C'_2)_6$	12	16	84	84¶
18	$(D_{6h}, C_1)_{24}$	1	1	4	
19	$(D_{6h}, C'_2)_{12}$	2	4	16	
20	$(T, C_1)_{12}$	2	2	12	
21	$(T, C_2)_6$	2	2	12	
22	$(T, C_3)_4$	2	2	12	12¶
23	$(T_h, C_1)_{24}$	1	1	7	
24	$(T_h, C_2)_{12}$	1	1	7	
25	$(T_h, C_3)_8$	1	1	7	7¶
26	$(T_h, C_s)_{12}$	1	1	7	
27	$(T_h, C_{2v})_6$	1	1	7	
28	$(O, C_1)_{24}$	3	3	24	
29	$(O, C_2)_{12}$	3	3	24	
30	$(O, C'_2)_{12}$	3	3	24	
31	$(O, C_3)_8$	3	3	24	
32	$(O, C_4)_6$	3	3	24	
33	$(O, D'_2)_6$	3	3	24	
34	$(O, D_3)_4$	3	3	24	24¶
35	$(O_h, C_1)_{48}$	1	1	10	
36	$(O_h, C_2)_{24}$	1	1	10	
37	$(O_h, C_s)_{24}$	1	1	10	
38	$(O_h, C'_2)_{24}$	2	2	20	
39	$(O_h, C_3)_{16}$	1	1	10	
40	$(O_h, C_4)_{12}$	2	2	20	
41	$(O_h, C_{2v})_{12}$	1	1	10	
42	$(O_h, D'_2)_{12}$	2	2	20	
43	$(O_h, C''_{2v})_{12}$	1	1	10	
44	$(O_h, D_3)_8$	2	2	20	20¶
45	$(O_h, C_{4v})_6$	2	2	20	
Total		244	279	2571	

* Zamorzaev (1953), Koptsik (1966).
† Zamorzaev *et al.* (1978).
‡ Harker (1981).
§ Koptsik & Kuzhukeev (1972).
¶ Karpova (1983).

as it must be, coincides with the number of the equi-translational 'epikernels' derived by Ascher (1977). This is another piece of evidence for the accuracy of this part of our list. Comparison of our tables with those of Ascher (1977) helped us to find some of Ascher's misprints: in his Table 5 it is necessary to interchange C_2^2 and C_2^3 in the D_2^5 column; in Table 8 C_{2h}^3 should be replaced by C_{2h}^6 in the D_{2h}^{18} column; in Table 14 D_2^6 should be replaced by D_2^5 in the D_6^6 column, and the second C_2 should be replaced by C'_s in the D_{3h} column.

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